

Self-Adjoint Extension of Symmetric Maps

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Abstract

A densely-defined symmetric linear map from/to a real Hilbert space extends to a self-adjoint map. Extension is expressed via Riesz representation. For a case including Friedrichs extension of a strongly monotone map, self-adjoint extension is unique, and equals closure of the given map.

Let $\{A : X \supseteq \mathcal{D}o(A) \rightarrow X\}$ be a densely-defined symmetric linear map. Recall that if Hilbert-space X is complex, then A may lack self-adjoint extension (see e.g. [R]). In contrast, self-adjoint extension must exist if our Hilbert-space is real, as will be shown here.

To prepare, we express well-known material in a form convenient for the present purpose. For $x \in X$, let $(x|A)$ denote the linear function $\{\mathcal{D}o(A) \ni y \mapsto (x|Ay)\}$; we use the convention that scalar-product is linear in the second entry, conjugate-linear in the first. Observe the adjoint domain $\mathcal{D}o(A^*)$ equals $\{x \in X : (x|A) \text{ continuous}\}$. Recall: $\mathcal{D}o(A) \subseteq \mathcal{D}o(A^*)$; A is self-adjoint iff $\mathcal{D}o(A) = \mathcal{D}o(A^*)$. Let J denote the duality-map on X , which maps x to function $(x|\cdot)$ in dual-space X^* ; write $J^{-1} = R$, Riesz-representation. Extend Riesz-map R so as to act on densely-defined (continuous linear) functions, such as $(x|A)$ if $x \in \mathcal{D}o(A^*)$.

Note. Let A have symmetric extension B . Then

- (i) $\mathcal{D}o(A) \subseteq \mathcal{D}o(B) \subseteq \mathcal{D}o(B^*) \subseteq \mathcal{D}o(A^*)$.
- (ii) $R(x|B) = R(x|A)$, if $x \in \mathcal{D}o(B^*) \subseteq \mathcal{D}o(A^*)$.
- (iii) $Bx = R(x|A)$ if $x \in \mathcal{D}o(B)$.

Proof. (i) is known. For $y \in \mathcal{D}o(A)$, see $(R(x|B)|y) = (x|By) = (x|Ay) = (R(x|A)|y)$; density of $\mathcal{D}o(A)$ gives (ii). For $x \in \mathcal{D}o(B)$ and $y \in \mathcal{D}o(A)$, see $(x|A)$ is continuous, and $(Bx|y) = (x|By) = (x|Ay) = (R(x|A)|y)$; density of $\mathcal{D}o(A)$ gives (iii). *Done.*

Denote by Λ the linear map $\{\mathcal{D}o(A^*) \ni x \mapsto R(x|A)\}$. *Note(iii)* (above) says A has at-most-one

symmetric extension to a given subspace Y , with $\mathcal{D}o(A) \subseteq Y \subseteq \mathcal{D}o(A^*)$; if such extension exists, then it equals the restriction $\Lambda|_Y$.

Theorem. Every symmetric map from/to a real Hilbert space has self-adjoint extension.

Proof. Let E denote the order-set of linear subspaces Y , with $\mathcal{D}o(A) \subseteq Y \subseteq \mathcal{D}o(A^*)$, for which restriction $\Lambda|_Y$ is symmetric; order by inclusion. ($E \ni \mathcal{D}o(A)$.) A chain C in E is bound above by the union of subspaces in C ; so Zorn's lemma ensures E has a maximal member, Z . $\Lambda|_Z$ is a maximal symmetric extension of A .

Write $\Lambda|_Z = M$. We claim $\mathcal{D}o(M) = \mathcal{D}o(M^*)$; if true, then M would be self-adjoint, concluding the proof. It is enough to show $\mathcal{D}o(M^*) \subseteq \mathcal{D}o(M)$; suppose not, seek a contradiction. Fix $p \in \mathcal{D}o(M^*) \setminus \mathcal{D}o(M)$. On the subspace $\mathcal{D}o(M) \oplus \mathbf{R}p$, define a map T :

$$T(x + ap) = Mx + aR(p|M) \text{ if } x \in \mathcal{D}o(M), a \in \mathbf{R}.$$

See T is linear, and T properly extends M . To show symmetry of T , let $\{x, y\} \subset \mathcal{D}o(M)$ and $\{a, b\} \subset \mathbf{R}$; note $(x|R(p|M)) = (p|Mx)$, $(R(p|M)|y) = (p|My)$; compute:

$$\begin{aligned} (T(x + ap)|y + bp) &= (Mx + aR(p|M)|y + bp) = \\ (Mx|y) + b(Mx|p) + a(R(p|M)|y) + ab(R(p|M)|p) &= \\ (x|My) + b(x|R(p|M)) + a(p|My) + ab(p|R(p|M)) &= \\ (x + ap|My + bR(p|M)) &= (x + ap|T(y + bp)). \end{aligned}$$

M has symmetric proper extension T , so M is not a maximal symmetric extension of A ; *contra*.

Done.

So, self-adjoint extension exists; now treat uniqueness. Fortunately, extension is unique for some cases of interest; sometimes we may even express extension simply, as closure of the given map. To prepare to show this, recall A has symmetric closure $\bar{A} \subseteq M$. Here, as before, $\{A : X \supseteq \mathcal{D}o(A) \rightarrow X\}$ is symmetric, with self-adjoint extension M , from/to a Hilbert space X , now assumed real. We also need the following two facts.

Note 1. If A has dense image and continuous inverse, then \bar{A} is the unique self-adjoint extension of A ; $M = \bar{A}$. \bar{A} maps onto X , and has continuous self-adjoint inverse.

Proof. \bar{A} has dense image (since A does); recall a symmetric map (\bar{A}) with dense image has symmetric inverse; \bar{A}^{-1} is also closed, since \bar{A} is so. \bar{A}^{-1} equals closure of a continuous map (A^{-1}), hence \bar{A}^{-1} is continuous. Since \bar{A}^{-1} is closed, continuous, and has dense domain (including $\mathcal{I}m(A)$), we have $\mathcal{D}o(\bar{A}^{-1}) = X$. A continuous symmetric map (\bar{A}^{-1}) on the whole Hilbert space is self-adjoint. Recall a self-adjoint map (\bar{A}^{-1}) with dense image (including $\mathcal{D}o(A)$) has self-adjoint inverse (\bar{A}). Hence $\{\bar{A}, M\}$ are self-adjoint extensions of A , with $\bar{A} \subseteq M$; this forces $\bar{A} = M$, because a self-adjoint map is maximal-symmetric. *Done.*

Note 2. A (densely-defined) closed 1:1 symmetric map has dense image.

Proof. It is enough to show $p = 0$, if $p \in \mathcal{Im}^\perp(A)$ (orthogonal complement of image). Since $\mathcal{Do}(A)$ is dense, it has a sequence $\{u_n\}$ converging to p . If $x \in \mathcal{Do}(A)$, then
 $0 = (p|Ax) = \lim(u_n|Ax) = \lim(Au_n|x)$. Density of $\mathcal{Do}(A)$ forces $\lim Au_n = 0$. A is closed; $(\lim u_n = p)$ and $(\lim Au_n = 0)$; hence $p \in \mathcal{Do}(A)$, $Ap = 0$. Since A is 1:1, we have $p = 0$. *Done.*

Recall (e.g. [Z]) that if our map A is strongly monotone, then it has Friedrichs extension, which is self-adjoint, 1:1, onto, with continuous self-adjoint inverse.

Theorem. If A is strongly monotone, then closure \bar{A} is the unique self-adjoint extension of A ; \bar{A} equals Friedrichs extension.

Proof. Let \hat{A} denote Friedrichs extension; $\hat{A} \supseteq \bar{A}$. Since \hat{A} is 1:1 with continuous inverse, so is its restriction \bar{A} . By *Note 2*, closed symmetric 1:1 map \bar{A} has dense image; then *Note 1* makes \bar{A} the unique self-adjoint extension of itself, and of A . \hat{A} is a self-adjoint extension of A , hence $\hat{A} = \bar{A}$.

Done.

Construction of the Friedrichs extension is complicated; how nice to express it simply (as closure), and to know it is the only self-adjoint extension.

References

- [R] Rudin, W. *Functional Analysis*. McGraw-Hill, 1991.
- [Z] Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics*. Springer, 1995.